

# A 2.75-Approximation Algorithm for the Unconstrained Traveling Tournament Problem

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**Abstract** A 2.75-approximation algorithm is proposed for the unconstrained traveling tournament problem, which is a variant of the traveling tournament problem. For the unconstrained traveling tournament problem, this is the first proposal of an approximation algorithm with a constant approximation ratio. In addition, the proposed algorithm yields a solution that meets both the no-repeater and mirrored constraints. Computational experiments show that the algorithm generates solutions of good quality.

**Keywords** timetable · sports scheduling · traveling tournament problem · approximation algorithm

## 1 Introduction

In the field of tournament timetabling, the traveling tournament problem (TTP) is a well-known benchmark problem established by Easton, Nemhauser, and Trick [4]. The present paper considers the unconstrained traveling tournament problem (UTTP), which is a variant of the TTP. In the following, some

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terminology and the TTP are introduced. The UTTP is then defined at the end of this section.

Given a set  $T$  of  $n$  teams, where  $n \geq 4$  and is even, a game is specified by an ordered pair of teams. Each team in  $T$  has its home venue. A double round-robin tournament is a set of games in which every team plays every other team once at its home venue and once as an away game (i.e., a game held at the home venue of the opponent). Consequently,  $2(n - 1)$  slots are necessary to complete a double round-robin tournament.

Each team stays at its home venue before a tournament and then travels to play its games at the chosen venues. After a tournament, each team returns to its home venue if the last game is played as an away game. When a team plays two consecutive away games, the team goes directly from the venue of the first opponent to the venue of another opponent without returning to its home venue.

For any pair of teams  $i, j \in T$ ,  $d_{ij} \geq 0$  denotes the distance between the home venues of  $i$  and  $j$ . Throughout the present paper, we assume that triangle inequality ( $d_{ij} + d_{jk} \geq d_{ik}$ ), symmetry ( $d_{ij} = d_{ji}$ ), and  $d_{ii} = 0$  hold for any teams  $i, j, k \in T$ .

Denote the distance matrix  $(d_{ij})$  by  $D$ . Given an integer parameter  $u \geq 2$ , the traveling tournament problem [4] is defined as follows.

**Traveling Tournament Problem (TTP( $u$ ))**

**Input:** A set of teams  $T$  and a distance matrix  $D = (d_{ij})$ .

**Output:** A double round-robin schedule of  $n$  teams such that

- C1. No team plays more than  $u$  consecutive away games,
- C2. No team plays more than  $u$  consecutive home games,
- C3. Game  $i$  at  $j$  immediately followed by game  $j$  at  $i$  is prohibited,
- C4. The total distance traveled by the teams is minimized.

Constraints C1 and C2 are referred to as the *atmost* constraints, and Constraint C3 is referred to as the *no-repeater* constraint.

Various studies on the TTP have been conducted in recent years (see [8, 10, 13] for detail), and most of these studies considered TTP(3) [14], which was recently proved to be NP-hard by Thielen and Westphal [12]. Almost all of the best upper bounds of TTP instances are obtained using metaheuristic algorithms. On the other hand, little research on approximation algorithms has been conducted for the TTP. Miyashiro, Matsui, and Imahori [9] proposed a  $(2 + O(1/n))$ -approximation algorithm for TTP(3). Yamaguchi, Imahori, Miyashiro, and Matsui [16] proposed an approximation algorithm for TTP( $u$ ), where  $3 \leq u \ll n$ . Westphal and Noparlik [15] proposed<sup>1</sup> a 5.875-approximation algorithm for TTP( $u$ ), where  $3 \leq u$ . For TTP(3), the approximation ratio of [16] is the best among them. In addition, Thielen and Westphal [11] proposed a  $(1.5 + O(1/n))$ -approximation algorithm for TTP(2).

<sup>1</sup> Westphal and Noparlik's paper [15] and the conference version of the present paper [7] appeared in the same conference (PATAT, 2010).

The TTP is a simplification of an actual sports scheduling problem. Some further simplified variants of the TTP have been studied [14]. The circular distance TTP and the constant distance TTP are the problems which have specific distance matrices. For the constant distance TTP, Fujiwara, Imahori, Matsui, and Miyashiro [5] proposed approximation algorithms.

The unconstrained traveling tournament problem (UTTP) is another variant of the TTP, in which Constraints C1 through C3 are eliminated. In other words, the UTTP is equivalent to  $\text{TTP}(n-1)$  without the no-repeater constraint. On some actual sports scheduling problems, the atmost constraints ( $u = 3$  in particular) and the no-repeater constraint are considered. However, these constraints are not necessarily imposed, and the UTTP is a suitable simplified model for some practical scheduling problems.

Bhattacharyya [1] recently showed NP-hardness of the UTTP. Although the UTTP is simpler than the TTP, no approximation algorithm has yet been proposed for the UTTP. The method proposed in [16] cannot be applied to the UTTP because the condition  $u \ll n$  is necessary. The method in [9], proposed for  $\text{TTP}(3)$ , can be applied to the UTTP with a few modifications. However, this leads to a  $((2/3)n + O(1))$ -approximation algorithm for the UTTP, which is not a constant approximation ratio with regard to  $n$ .

In the present paper, we propose a 2.75-approximation algorithm for the UTTP. In addition, the solution obtained by the algorithm meets both the no-repeater and mirrored constraints, which are sometimes required in practice. This property indicates that our algorithm also works for  $\text{TTP}(n-1)$ , which eliminates the atmost constraints but considers the no-repeater constraint.

## 2 Algorithm

In this section, we propose an approximation algorithm for the UTTP. A key concept of the algorithm is the use of the circle method and a shortest Hamilton cycle. The classical schedule obtained by the circle method satisfies the property that, for all teams but one, the orders of opponents are very similar to a mutual cyclic order. Roughly speaking, the proposed algorithm constructs a short Hamilton cycle passing all venues, and finds a permutation of teams such that the above cyclic order corresponds to the Hamilton cycle.

Let  $G = (V, E)$  be a complete undirected graph with the vertex set  $V$  and edge set  $E$ , where  $|V| = n$ . We assume that there exists a bijection between the vertex set  $V$  and the set of teams  $T$ . We put the length of edge  $\{v, v'\} \in E$ , denoted by  $d_{vv'}$ , to the distance between the home venues of the corresponding teams  $t, t' \in T$ . First, we assign aliases  $0, 1, \dots, n-1$  to teams in  $T$  as follows.

1. For each  $v \in V$ , compute  $\sum_{v' \in V \setminus \{v\}} d_{vv'}$ .
2. Let  $v^*$  be a vertex that attains  $\min_{v \in V} \sum_{v' \in V \setminus \{v\}} d_{vv'}$ , and designate the team corresponding to  $v^*$  as team  $n-1$ .
3. Using Christofides' 1.5-approximation algorithm [2] for the traveling salesman problem with triangle inequality and symmetry, construct a Hamilton

cycle on the complete graph induced by  $V \setminus \{v^*\}$ . For the obtained cycle  $(v_0, v_1, \dots, v_{n-2})$ , denote the corresponding teams by  $(0, 1, \dots, n-2)$ .

In the rest of this paper, we define that the set of teams  $T = \{0, 1, 2, \dots, n-1\}$  and the vertex set  $V = \{v_0, v_1, \dots, v_{n-2}, v^*\}$ . We identify the vertex  $v_{n-1}$  with  $v_0$  (not  $v^*$ ) and the vertex  $v_{-1}$  with  $v_{n-2}$  (not  $v^*$ ).

Next, we construct a single round-robin schedule. In the following, a “schedule without HA-assignment” refers to a “round-robin schedule without the concepts of home game, away game, and venue.” Denote the set of  $n-1$  slots by  $S = \{0, 1, \dots, n-2\}$ . A single round-robin schedule without HA-assignment is a matrix  $K$  of which  $(t, s) \in T \times S$  element, say  $K(t, s)$ , denotes the opponent of team  $t$  in slot  $s$ . Let  $K^*$  be a matrix defined by

$$K^*(t, s) = \begin{cases} s - t \pmod{n-1} & (t \neq n-1 \text{ and } s - t \not\equiv t \pmod{n-1}), \\ n-1 & (t \neq n-1 \text{ and } s - t \equiv t \pmod{n-1}), \\ s/2 & (t = n-1 \text{ and } s \text{ is even}), \\ (s+n-1)/2 & (t = n-1 \text{ and } s \text{ is odd}). \end{cases}$$

**Lemma 1** [16] *The matrix  $K^*$  is a single round-robin schedule without HA-assignment. In addition,  $K^*$  is essentially equivalent to the classical schedule obtained by the circle method.*

Then, by the mirroring procedure, we form  $K^*$  into a double round-robin schedule without HA-assignment. More precisely, construct a matrix  $(K^*|K^*)$  whose rows are index by teams and columns are index by a sequence of slots  $(0, 1, \dots, n-2, n-1, n, \dots, 2n-3)$ . So as to complete a double round-robin schedule, “home” and “away” are assigned to games of  $(K^*|K^*)$  as follows:

- for team  $t \in \{0, 1, \dots, n/2-1\}$ , let the games in slots  $2t, 2t+1, \dots, n+2t-2$  be home games, and let the other games be away games.
- for team  $t \in \{n/2, n/2+1, \dots, n-2\}$ , let the games in slots  $2t-n+2, 2t-n+3, \dots, 2t$  be away games, and let the other games be home games.
- for team  $n-1$ , let the games in slots  $0, 1, \dots, n-2$  be away games, and let the other games be home games.

The obtained double round-robin schedule is denoted by  $K_{\text{DRR}}^*$ . Figure 1 shows the schedule  $K_{\text{DRR}}^*$  of 10 teams.

**Lemma 2** *The double round-robin schedule  $K_{\text{DRR}}^*$  is feasible.*

**Proof.**  $(K^*|K^*)$  is a consistent double round-robin schedule without HA-assignment, which satisfies the mirrored constraint. We check the feasibility of HA-assignment to games. Teams  $i$  and  $j$  ( $i < j < n-1$ ) have a game at slot  $i+j$ . By the rule to assign home and away to games, team  $i$  plays a home game and team  $j$  plays an away game at slot  $i+j$ . Teams  $i$  and  $j$  ( $i < j = n-1$ ) have a game at slot  $2i$ , and the rule assigns consistent home/away to the teams. Another game between teams  $i$  and  $j$  is held at the opposite venue.  $\square$

slots teams	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	9H	1H	2H	3H	4H	5H	6H	7H	8H	9A	1A	2A	3A	4A	5A	6A	7A	8A
1	8A	0A	9H	2H	3H	4H	5H	6H	7H	8H	0H	9A	2A	3A	4A	5A	6A	7A
2	7A	8A	0A	1A	9H	3H	4H	5H	6H	7H	8H	0H	1H	9A	3A	4A	5A	6A
3	6A	7A	8A	0A	1A	2A	9H	4H	5H	6H	7H	8H	0H	1H	2H	9A	4A	5A
4	5A	6A	7A	8A	0A	1A	2A	3A	9H	5H	6H	7H	8H	0H	1H	2H	3H	9A
5	4H	9H	6A	7A	8A	0A	1A	2A	3A	4A	9A	6H	7H	8H	0H	1H	2H	3H
6	3H	4H	5H	9H	7A	8A	0A	1A	2A	3A	4A	5A	9A	7H	8H	0H	1H	2H
7	2H	3H	4H	5H	6H	9H	8A	0A	1A	2A	3A	4A	5A	6A	9A	8H	0H	1H
8	1H	2H	3H	4H	5H	6H	7H	9H	0A	1A	2A	3A	4A	5A	6A	7A	9A	0H
9	0A	5A	1A	6A	2A	7A	3A	8A	4A	0H	5H	1H	6H	2H	7H	3H	8H	4H

Each number corresponds to the opponent and away (home) game is denoted by A (H).

**Fig. 1** The schedule  $K_{\text{DRR}}^*$  with 10 teams.

In addition, for each  $m \in \{0, 1, \dots, 2n-3\}$ , we construct a double round-robin schedule by rotating slots of  $K_{\text{DRR}}^*$  through  $m$  cyclically. It means that games of  $K_{\text{DRR}}^*(m)$  ( $m \in \{0, 1, \dots, 2n-3\}$ ) at slot  $s$  are equal to games of  $K_{\text{DRR}}^*$  at slot  $s+m \pmod{2n-2}$ . Obviously, all of the schedules  $K_{\text{DRR}}^*(m)$  ( $m \in \{0, 1, \dots, 2n-3\}$ ) meet both the no-repeater and mirrored constraints. Finally, output a best solution among  $K_{\text{DRR}}^*(m)$  ( $m \in \{0, 1, \dots, 2n-3\}$ ).

Here, we estimate the time complexity of the algorithm. Christofides' algorithm requires  $O(n^3)$  time to construct a Hamilton cycle on the complete graph induced by  $V \setminus \{v^*\}$ . For the constructed Hamilton cycle, there are  $2(n-1)$  possibilities to assign teams. For each assignment of teams, we consider  $2n-2$  possibilities of  $m \in \{0, 1, \dots, 2n-3\}$ . Each double round-robin schedule can be evaluated in  $O(n)$  time on average. Thus, the time complexity of the algorithm is bounded by  $O(n^3)$ .

In the next section, we prove that the proposed algorithm guarantees an approximation ratio 2.75.

### 3 Approximation Ratio

In this section, we describe the proof of the approximation ratio of the proposed algorithm. Designate the length of a shortest Hamilton cycle on  $G$  as  $\tau$ .

**Lemma 3** *The following propositions hold for  $G$ .*

- (1) *The length of any edge is bounded by  $\tau/2$ .*
- (2) *The length of any Hamilton cycle on  $G$  is bounded by  $n\tau/2$ .*
- (3) 
$$\sum_{v \in V} \sum_{v' \in V \setminus \{v\}} d_{vv'} \leq n^2\tau/4.$$
- (4) 
$$\sum_{v \in V \setminus \{v^*\}} d_{vv^*} \leq n\tau/4.$$

**Proof.** (1) For the edges  $\{i, j\}$  and  $\{j, i\}$ , the sum of their lengths is at most the length of a shortest Hamilton cycle. Thus, the length of the edge  $\{i, j\}$  is bounded by  $\tau/2$  with symmetry.

- (2) This immediately follows from Property (1).  
 (3) Given a shortest Hamilton cycle  $H = (u_0, u_1, \dots, u_{n-1})$  on  $G$ , let

$$h_{u_i, u_j} = \begin{cases} d_{u_i, u_{i+1}} + d_{u_{i+1}, u_{i+2}} + \dots + d_{u_{j-1}, u_j} & (j - i \pmod{n} \leq n/2), \\ d_{u_i, u_{i-1}} + d_{u_{i-1}, u_{i-2}} + \dots + d_{u_{j+1}, u_j} & (j - i \pmod{n} > n/2). \end{cases}$$

Then, we have:

$$\begin{aligned} \sum_{v \in V} \sum_{v' \in V \setminus \{v\}} d_{vv'} &= \sum_{i=0}^{n-1} \sum_{k=1}^{n-1} d_{u_i, u_{i+k \pmod{n}}} \\ &\leq \sum_{k=1}^{n-1} \sum_{i=0}^{n-1} h_{u_i, u_{i+k \pmod{n}}} \\ &= \sum_{k=1}^{n/2-1} \sum_{i=0}^{n-1} (d_{u_i, u_{i+1}} + d_{u_{i+1}, u_{i+2}} + \dots + d_{u_{i+k-1}, u_{i+k}}) \\ &\quad + \sum_{k=n/2+1}^{n-1} \sum_{i=0}^{n-1} (d_{u_i, u_{i-1}} + d_{u_{i-1}, u_{i-2}} + \dots + d_{u_{i-n+k+1}, u_{i-n+k}}) \\ &\quad + \sum_{i=0}^{n-1} (d_{u_i, u_{i+1}} + d_{u_{i+1}, u_{i+2}} + \dots + d_{u_{i+n/2-1}, u_{i+n/2}}) \\ &= 2 \left( 1 + 2 + \dots + \left( \frac{n}{2} - 1 \right) \right) \tau + \frac{n\tau}{2} = \frac{n^2\tau}{4}. \end{aligned}$$

- (4) Since  $v^*$  is a vertex that attains  $\min_{v \in V} \sum_{v' \in V \setminus \{v\}} d_{vv'}$ , the inequality obtained in (3) directly implies the desired one.  $\square$

Now we discuss the average of the traveling distances of  $K_{\text{DRR}}^*(m)$  ( $m \in \{0, 1, \dots, 2n-3\}$ ). The traveling distance of a schedule is subject to the following constraint, say the *athome* constraint: each team stays at its home venue before a tournament and returns to its home venue after a tournament. For simplicity of the analysis of the approximation ratio, we temporary replace the *athome* constraint with the following assumption.

**Assumption A.** If a team plays away games at both the first and last slots, then the team moves from the venue of the last opponent to that of the first opponent, instead of the moves before the first slot and after the last slot.

We discuss a traveling distance of each team under Assumption A. Application of Assumption A guarantees that a route of each team in  $K_{\text{DRR}}^*(m)$  ( $m \in \{0, 1, \dots, 2n-3\}$ ) is a Hamilton cycle on  $G$  (see Figure 2), and the traveling distance of  $K_{\text{DRR}}^*(m)$  is invariant with respect to  $m \in \{0, 1, \dots, 2n-3\}$ . Thus, we only need to consider  $K_{\text{DRR}}^*$ . This assumption makes the analysis of the approximation ratio much easier.

Let the length of the cycle  $(v_0, v_1, \dots, v_{n-2})$  obtained by Christofides' method in the proposed algorithm be  $\tau'$ . Note that  $\tau' \leq (3/2)\tau$ , where  $\tau$  denotes the length of a shortest Hamilton cycle on  $G$ . Analyzing the structure of  $K_{\text{DRR}}^*$  reveals the following lemma.

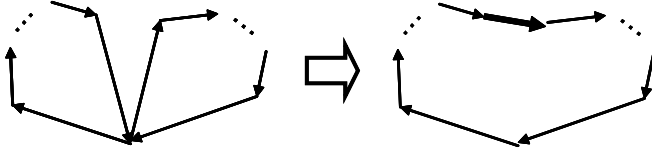


Fig. 2 Effect of Assumption A.

**Lemma 4** *Under Assumption A, the traveling distance of team  $t$  in  $K_{\text{DRR}}^*$  is bounded by*

$$\begin{cases} \tau' + d_{v_t, v^*} + d_{v^*, v_{t+1}} - d_{v_t, v_{t+1}} & (t \in \{0, 1, \dots, n/2 - 1\}), \\ \tau' + d_{v_{t-1}, v^*} + d_{v^*, v_t} - d_{v_{t-1}, v_t} & (t \in \{n/2, n/2 + 1, \dots, n - 2\}), \\ n\tau/2 & (t = n - 1). \end{cases}$$

**Proof.** When  $t \in \{0, 1, 2, \dots, n/2 - 1\}$ , team  $t$  moves along a Hamilton cycle  $(v_t, v^*, v_{t+1}, \dots, v_{n-2}, v_0, v_1, v_2, \dots, v_{t-1})$ . Consequently, the length of the tour is  $\tau' + d_{v_t, v^*} + d_{v^*, v_{t+1}} - d_{v_t, v_{t+1}}$ .

When  $t \in \{n/2, n/2 + 1, \dots, n - 2\}$ , a tour of team  $t$  is a Hamilton cycle  $(v_t, v_{t+1}, \dots, v_{n-2}, v_0, v_1, v_2, \dots, v_{t-1}, v^*)$ , and thus the length is  $\tau' + d_{v_{t-1}, v^*} + d_{v^*, v_t} - d_{v_{t-1}, v_t}$ .

Since a tour of team  $n - 1$  is Hamiltonian, Lemma 3(2) implies the desired result.  $\square$

The above lemma implies an upper bound of the traveling distance of  $K_{\text{DRR}}^*$ .

**Lemma 5** *Under Assumption A, the traveling distance of  $K_{\text{DRR}}^*$  is bounded by  $(n - 2)\tau' + 2 \sum_{v \in V \setminus \{v^*\}} d_{vv^*} + (3/2)\tau + n\tau/2$ .*

**Proof.** Consider the sum total of upper bounds obtained in Lemma 4

$$(n - 1)\tau' + L + n\tau/2$$

where

$$\begin{aligned} L &\stackrel{\text{def.}}{=} \sum_{t \in \{0, 1, \dots, n/2 - 1\}} (d_{v_t, v^*} + d_{v^*, v_{t+1}} - d_{v_t, v_{t+1}}) \\ &\quad + \sum_{t \in \{n/2, n/2 + 1, \dots, n - 2\}} (d_{v_{t-1}, v^*} + d_{v^*, v_t} - d_{v_{t-1}, v_t}). \end{aligned}$$

It is easy to see that

$$\begin{aligned}
L &= \left( \sum_{t \in \{0,1,\dots,n/2-1\}} d_{v_t, v^*} \right) + \left( \sum_{t \in \{1,2,\dots,n/2\}} d_{v_t, v^*} \right) \\
&\quad - \left( \sum_{t \in \{0,1,\dots,n/2-1\}} d_{v_t, v_{t+1}} \right) + \left( \sum_{t \in \{n/2-1, n/2, \dots, n-3\}} d_{v_t, v^*} \right) \\
&\quad + \left( \sum_{t \in \{n/2, n/2+1, \dots, n-2\}} d_{v_t, v^*} \right) - \left( \sum_{t \in \{n/2-1, n/2, \dots, n-3\}} d_{v_t, v_{t+1}} \right) \\
&\leq 2 \sum_{v \in V \setminus \{v^*\}} d_{vv^*} - \sum_{t \in \{0,1,\dots,n-2\}} d_{v_t, v_{t+1}} + d_{v_{n/2-1}, v^*} + d_{v_{n/2}, v^*} + d_{v_{n-2}, v_0} \\
&\leq 2 \sum_{v \in V \setminus \{v^*\}} d_{vv^*} - \tau' + (3/2)\tau
\end{aligned}$$

where the last inequality follows from Lemma 3(1). From the above, the lemma holds.  $\square$

Here we drop Assumption A and restore the at-home constraint, and consider the increase of the traveling distance in the following lemma.

**Lemma 6** *For each team  $t$ , let  $\ell_A(t)$  be the traveling distance of  $t$  in  $K_{\text{DRR}}^*$  under Assumption A. Then, with the at-home constraint the average of the traveling distances of team  $t$  among  $K_{\text{DRR}}^*(m)$  ( $m \in \{0, 1, \dots, 2n-3\}$ ) is bounded by  $\ell_A(t) + \sum_{v' \in V \setminus \{v\}} d_{vv'}/(n-1)$ , where  $v$  is the home venue of  $t$ .*

**Proof.** For a choice  $m \in \{0, 1, \dots, 2n-3\}$ , every team  $t'$  different from  $t$  plays away game with  $t$  at first slot just once. Thus, the average length of the moves of team  $t$  before the first slot is bounded by  $\sum_{v' \in V \setminus \{v\}} d_{vv'}/(2n-2)$ . Similarly, the average length of the moves of team  $t$  after the last slot is bounded by  $\sum_{v' \in V \setminus \{v\}} d_{vv'}/(2n-2)$ . Thus, the average of the traveling distances of team  $t$  is bounded by  $\ell_A(t) + \sum_{v' \in V \setminus \{v\}} d_{vv'}/(n-1)$ .  $\square$

Summarizing the above lemmas, we have the following theorem.

**Theorem 1** *The average of the total traveling distances of schedules  $K_{\text{DRR}}^*(m)$  ( $m \in \{0, 1, \dots, 2n-3\}$ ) is bounded by*

$$(n-2)\tau' + 2 \sum_{v \in V \setminus \{v^*\}} d_{vv^*} + (3/2)\tau + n\tau/2 + \sum_{v \in V} \sum_{v' \in V \setminus \{v\}} d_{vv'}/(n-1).$$

Lastly we show the approximation ratio of the proposed algorithm.

**Theorem 2** *The proposed algorithm is a 2.75-approximation algorithm for the UTPP.*

**Proof.** Let  $z^*$  be the average of the total traveling distances of schedules



$K_{\text{DRR}}^*(m)$  ( $m \in \{0, 1, \dots, 2n - 3\}$ ). From Theorem 1 and Lemma 3(3)(4), we have:

$$\begin{aligned}
z^* &\leq (n-2)\tau' + 2 \sum_{v \in V \setminus \{v^*\}} d_{vv^*} + (3/2)\tau + n\tau/2 + \sum_{v \in V} \sum_{v' \in V \setminus \{v\}} d_{vv'}/(n-1) \\
&\leq (n-2)(3/2)\tau + 2n\tau/4 + (3/2)\tau + n\tau/2 + (n^2\tau/4)/(n-1) \\
&= (3/2)n\tau - 3\tau + (1/2)n\tau + (3/2)\tau + (1/2)n\tau + (1/4)n\tau + (1/4)n\tau/(n-1) \\
&= (11/4)n\tau - (3/2)\tau + (1/4)n\tau/(n-1) \leq (11/4)n\tau.
\end{aligned}$$

The proposed algorithm output a best of  $K_{\text{DRR}}^*(m)$  ( $m \in \{0, 1, \dots, 2n - 3\}$ ), and thus the traveling distance of the output is at most  $z^*$ . Since  $n\tau$  is a lower bound of the distance of any double round-robin schedule, this concludes the proof.  $\square$

Let us consider a case that we have a shortest Hamilton cycle  $H$  on  $G$ . In this situation, the following corollary holds.

**Corollary 1** *If a shortest Hamilton cycle  $H$  on  $G$  is given, there exists a 2.25-approximation algorithm for the UTTP.*

**Proof.** We replace a cycle obtained by Christofides' method in the proposed algorithm with a cycle obtained from  $H$  by skipping vertex  $v^*$ . Theorem 1 implies that the average of total traveling distances of schedules, say  $z^{**}$ , obtained by the proposed algorithm is bounded by

$$\begin{aligned}
z^{**} &\leq (n-2)\tau + 2 \sum_{v \in V \setminus \{v^*\}} d_{vv^*} + (3/2)\tau + n\tau/2 + \sum_{v \in V} \sum_{v' \in V \setminus \{v\}} d_{vv'}/(n-1) \\
&\leq n\tau - 2\tau + 2n\tau/4 + (3/2)\tau + n\tau/2 + (1/4)n\tau + (1/4)n\tau/(n-1) \\
&= (9/4)n\tau - \tau/2 + (1/4)n\tau/(n-1) \leq (9/4)n\tau.
\end{aligned}$$

Thus, the approximation ratio is bounded by 2.25 in this case.  $\square$

## 4 Computational Results

In this section, we describe the results of computational experiments using the proposed approximation algorithm.

For the experiments, we took the distance matrices of NL and galaxy instances from the website [14], because they are the most popular instances and one having the largest distance matrix (up to 40 teams), respectively. We ran the proposed algorithm for the UTTP version of these instances; to find a short Hamilton cycle, we use Concorde TSP solver [3]. It took less than one second to obtain a shortest Hamilton cycle even for the largest case ( $n = 40$ ).

To evaluate the quality of obtained solutions, we also tried to find optimal solutions of UTTP instances with integer programming. Computations using integer programming were performed on the following PC: Intel Xeon 3.33GHz\*2, 24GB RAM, Windows 7 64bit, and Gurobi Optimizer 4.5.1 [6] with 16 threads as an integer programming solver. For both NL and galaxy

**Table 1** Results for the UTTP version of NL instances

$n$	approx.	$n * \text{TSP}$	gap (%) <sup>†</sup>	best UB
4	8276	8044	2.9	8276 <sup>‡</sup>
6	20547	17826	15.3	19900 <sup>‡</sup>
8	33190	27840	19.2	30700 <sup>‡</sup>
10	47930	38340	25.0	45605 <sup>*</sup>
12	81712	67200	21.6	
14	128358	103978	23.4	
16	156828	119088	31.7	

<sup>†</sup>gap is obtained by  $(\frac{\text{approx.}}{n * \text{TSP}} - 1) * 100.0$

<sup>‡</sup>optimal

<sup>\*</sup>best incumbent solution after 500,000 seconds

instances: for  $n = 4, 6, 8$  optimal solutions were obtained; for  $n = 10$ , after 500,000 seconds of computations, branch-and-bound procedures were not terminated; for  $n = 12$  and larger instances, using integer programming it was difficult to find solutions better than those obtained by the proposed algorithm.

Tables 1 and 2 show the results of experiments for NL and galaxy instances, respectively. The first columns of tables denote the number of teams,  $n$ . The second ones are the total traveling distance obtained by the proposed algorithm. The third ones are the value of  $n$  times the distance of a shortest Hamilton cycle, as a simple lower bound. The fourth ones are the percentages of the gap between the second and third columns.

Like most theoretical approximation algorithms, the obtained gaps are much better than the theoretical approximation ratio 2.75 (175% gap). For the NL instances and the galaxy instances of up to 20 teams, the gap is around 25%. For the galaxy instances of more than 20 teams, the gap is less than 20%. Note that the gaps shown in the tables are from the ratio of the obtained distance to a lower bound, but not to optimal distance. Therefore the gaps between the obtained distance and the optimal value are still better than the gaps shown in the tables.

## 5 Conclusion

This paper proposed an approximation algorithm for the unconstrained traveling tournament problem, which is a variant of the traveling tournament problem. The approximation ratio of the proposed algorithm is 2.75, and the algorithm yields a solution satisfying the no-repeater and mirrored constraints. If a shortest Hamilton cycle on the home venues of the teams is available, the approximation ratio is improved to 2.25. Computational experiments showed that the algorithm generates solutions of good quality; the gap between the obtained solution and a simple lower bound is around 25% for small instances (up to 20 teams) and is less than 20% for larger instances.

**Table 2** Results for the UTPP version of galaxy instances

$n$	approx.	$n * \text{TSP}$	gap (%) <sup>†</sup>	best UB
4	416	412	1.0	416 <sup>‡</sup>
6	1197	1068	12.1	1178 <sup>‡</sup>
8	2076	1672	24.2	1890 <sup>‡</sup>
10	3676	3020	21.7	3570*
12	5514	4524	21.9	
14	7611	6216	22.4	
16	9295	7408	25.5	
18	12320	10026	22.9	
20	14739	11880	24.1	
22	19525	16522	18.2	
24	25026	21216	18.0	
26	32250	27846	15.8	
28	41843	36708	14.0	
30	52073	46410	12.2	
32	62093	55104	12.7	
34	77392	69326	11.6	
36	88721	78624	12.8	
38	103988	92568	12.3	
40	120895	107800	12.1	

<sup>†</sup>gap is obtained by  $(\frac{\text{approx.}}{n * \text{TSP}} - 1) * 100.0$

<sup>‡</sup>optimal

\*best incumbent solution after 500,000 seconds

## References

1. Bhattacharyya, R., A note on complexity of traveling tournament problem, available from Optimization Online, 2009  
Web page <http://www.optimization-online.org/>
2. Christofides, N., Worst-case analysis of a new heuristic for the traveling salesman problem, Report 388, Graduate School of Industrial Administration, Carnegie-Mellon University, 1976
3. Cook, W., Concorde TSP solver, Web page, as of 2011  
<http://www.tsp.gatech.edu/concorde.html>
4. Easton, K., Nemhauser, G., Trick, M., The traveling tournament problem: description and benchmarks, *Lecture Notes in Computer Science*, **2239**, 580–584 (2001)
5. Fujiwara, N., Imahori, S., Matsui, T., Miyashiro, R., Constructive algorithms for the constant distance traveling tournament problem, *Lecture Notes in Computer Science*, **3867**, 135–146 (2007)
6. Gurobi Optimization, Gurobi Optimizer 4.5.1. (2011)  
<http://www.gurobi.com/>
7. Imahori, S., Matsui, T., Miyashiro, R., An approximation algorithm for the unconstrained traveling tournament problem, In: Proceedings of the 8th International Conference on the Practice and Theory of Automated Timetabling (PATAT), 508–512 (2010)
8. Kendall, G., Knust, S., Ribeiro, C.C., Urrutia, S., Scheduling in sports: an annotated bibliography, *Computers and Operations Research*, **37**, 1–19 (2010)
9. Miyashiro, R., Matsui, T., Imahori, S., An approximation algorithm for the traveling tournament problem, *Annals of Operations Research*, to appear
10. Rasmussen, R.V., Trick, M.A., Round robin scheduling — a survey, *European Journal of Operational Research*, **188**, 617–636 (2008)
11. Thielen, C., Westphal, S., Approximating the traveling tournament problem with maximum tour length 2, *Lecture Notes in Computer Science*, **6507**, 303–314 (2010)

- 
12. Thielen, C., Westphal, S., Complexity of the traveling tournament problem, *Theoretical Computer Science*, **412**, 345–351 (2011)
  13. Trick, M.A., Sports Scheduling, Hybrid Optimization (Springer Optimization and Its Applications, 2011, Volume 45), 489–508 (2011)
  14. Trick, M., Challenge traveling tournament problems, Web page, as of 2010  
<http://mat.gsia.cmu.edu/TOURN/>
  15. Westphal, S., Noparlik, K., A 5.875-approximation for the traveling tournament problem, In: Proceedings of the 8th International Conference on the Practice and Theory of Automated Timetabling (PATAT), 417–426 (2010)
  16. Yamaguchi, D., Imahori, S., Miyashiro, R., Matsui, T., An improved approximation algorithm for the traveling tournament problem, *Algorithmica*, to appear